# An Equilibrium Model of Cryptocurrencies

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## 1 Introduction

The cryptocurrencies' prices fluctuate dramatically, especially when traditional financial market experiences crash, like in middle March in 2020. What are the drivers of price fluctuation? This paper studies the demand and supply of cryptocurrencies and establish an equilibrium model of cryptocurrency price under rational expectations.

There are two pieces of literature closely related to this paper. One is Cao et al. (2020), in which authors provide a one-period equilibrium model with speculative demand. The other one is Jermann (2018), in which the author offers a multi-period equilibrium based on Cagan's model of hyperinflation, but doesn't model the speculative demand.

## 2 Model

### 2.1 Model Setup

For one cryptocurrency, the total supply is

$$C_t^s = S_t \cdot M_t \tag{1}$$

where $S_t$ is the price of the cryptocurrency denominated by USD and $M_t$ is is total stock of the cryptocurrency.

The demand can be broken down into two forces: exchange and speculation. The demand of cryptocurrency as medium of exchange follows Barro (1979),

$$C^d_{Exchange,t} = \frac{P_t G_t}{V_t} \tag{2}$$

where $P_t$ is the price level, $G_t$ is the size of the cryptocurrency economy, and $V_t$ is the velocity of the cryptocurrency.

The demand for speculation follows Cao et al. (2020),

$$C^d_{Speculation,t} = f(S_t, A_t) \tag{3}$$

$A_t$ is investment attractiveness of the cryptocurrency. It is depends on current price of the cryptocurrency, and attractiveness as well. There are two key assumptions underlying further modeling.

**Assumption 1:** As in Cao et al. (2020), $f$ is assumed to be increasing in $A$, and deceasing in $S$, with $\frac{\partial f}{\partial S}$ increases in $S$ and $\frac{\partial f}{\partial S} \to 0$ as $S \to \infty$

**Assumption 2:** There is purchasing power parity for cryptocurrency with USD, i.e. $S_t = \frac{P^{USD}_t}{P_t}$. Furthermore, USD is the numeraire in this paper. Then,

$$S_t = \frac{1}{P_t} \tag{4}$$

## 2.2 Equilibrium

To make equilibrium more tackleable, we propose that the demand for speculation is proportional to the demand for exchange, i.e.

$$C^d_{Speculation,t} = f(S_t, A_t) = g(S_t, A_t) \cdot C^d_{Exchange,t} = \frac{g(S_t, A_t)}{S_t}\frac{G_t}{V_t} \tag{5}$$

which means there are more speculative activities when more demand for cryptocurrency as medium of exchange.

In the equilibrium, the supply $C^s_t$ equals the demand $C^d_t$ for each $t$:

$$S_t \cdot M_t = \frac{P_t G_t}{V_t}\left(1 + g(S_t, A_t)\right) \tag{6}$$

To illustrate, we assume that

$$g(S_t, A_t) = K \cdot S_t^{\eta_1} \cdot A_t^{\eta_2}, with\ \eta_1 < 1, \eta_2 > 0 \tag{7}$$

This function imposes productive separability between attractiveness $A_t$ and price $S_t$, and a constant inter-temporal elasticity of substitution for $A_t$ and $S_t$. With $\eta_1 < 1, \eta_2 > 0$, the speculative demand function satisfies Assumption 1.

## 2.3 Log Linearization

Writing all variables in natural logarithm terms,

$$c_t^s = s_t + m_t \tag{8}$$

$$c_t^d = c_{Exchange,t}^d + c_{Speculation,t}^d = -s_t + g_t - v_t + \log(1 + e^{k+\eta_1 s_t + \eta_2 a_t}) \tag{9}$$

where all small letters represent the log of responding capital letters.

By $\log(1+e^x) = x + \log(1+e^{-x})$, we have $\log(1+e^{k+\eta_1 s_t+\eta_2 a_t}) = k + \eta_1 s_t + \eta_2 a_t + \log(1+e^{-(k+\eta_1 s_t+\eta_2 a_t)})$. Therefore, under the equilibrium, the price $s_t$ satisfies

$$s_t + m_t = k + (\eta_1 - 1)s_t + \eta_2 a_t + g_t - v_t + \log(1 + e^{-(k+\eta_1 s_t+\eta_2 a_t)}) \tag{10}$$

**Proposition 1 :** Using implied function theorem, we have the sensitivity of equilibrium price $s_t$ with respect to attractiveness $a_t$,

$$\frac{\partial s_t}{\partial a_t} = -\frac{\partial f/\partial a_t}{\partial f/\partial s_t} = \frac{\eta_2(1 - \frac{1}{e^{k+\eta_1 s_t+\eta_2 a_t}+1})}{2 - \eta_1(1 - \frac{1}{e^{k+\eta_1 s_t+\eta_2 a_t}+1})} > 0 \tag{11}$$

where $f(s_t, a_t) = k + (\eta_1 - 2)s_t + \eta_2 a_t + g_t - v_t + \log(1 + e^{-(k+\eta_1 s_t+\eta_2 a_t)}) - m_t = 0$. So the sensitivity dependents on the current level of $s_t$ and $a_t$.

## 2.4 Velocity and Attractiveness

Up to now, we have established the equilibrium for each time period $t$. Under the rational expectation framework, we could have the dynamic of equilibrium price to see how the price evolves over time.

As $v_t$ in the Cagan's model of hyperinflation,

$$v_t = \alpha(\mathbb{E}_t p_{t+1} - p_t) + \gamma_t \tag{12}$$

where $\gamma_t$ is the velocity shock caused by technology development, $(\mathbb{E}_t p_{t+1} - p_t)$ is expected inflation and the elasticity of velocity with respect to inflation $\alpha > 0$. Follow Jermann (2018), under the Assumption 2,

$$v_t = -\alpha(\mathbb{E}_t s_{t+1} - s_t) + \gamma_t \tag{13}$$

In this case, under the equilibrium

$$(2-\eta_1+\alpha)\, s_t = \alpha \mathbb{E}_t s_{t+1} + k + \eta_2 a_t + g_t - m_t - \gamma_t + \log(1+e^{-(k+\eta_1 s_t+\eta_2 a_t)}) \tag{14}$$

For $a_t$, we purpose a similar model,

$$a_t = \beta(\mathbb{E}_t s_{t+1} - s_t) + \theta g_t + \omega_t \tag{15}$$

where $\omega_t$ is regulatory shock, which could be either positive or negative. $(\mathbb{E}_t s_{t+1} - s_t)$ is expected appreciation and $\beta > 0$, $\theta > 0$ .

**Proposition 2 :** If the dynamic of $v_t$ and $a_t$ follows Equation (13) and (15) respectively, under the equilibrium, the market price $s_t$ satisfies

$$\begin{aligned}(2-\eta_1+\eta_2\beta+\alpha)\, s_t = {} & (\eta_2\beta+\alpha)\mathbb{E}_t s_{t+1} + k + (1+\eta_2\theta) g_t - m_t + \eta_2\omega_t - \gamma_t \\ & + \exp\{-[k+(\eta_1-\eta_2\beta)s_t + \eta_2\beta\mathbb{E}_t s_{t+1} + \eta_2\theta g_t + \eta_2\omega_t]\}\end{aligned} \tag{16}$$

### 2.5 Numerical Illustration

Assume the size of cryptocurrency economy is stable, i.e. $G_t = 1$ for any $t$. The stock of cryptocurrency has geometric growth, i.e. $M_t = M_0(1+r)^t$. For example, the first-year size $M_0$ of Bitcoin is 12.5% of total Bitcoin mineable and compound annual growth rate $r$ is approximately 0.125 up to 2019. In numerical part, we take those two number as in Bitcoin. By Equation (16), we can simulate the path of price for $T = 10$ under different parameters in $a_t$ and $v_t$.

We first assume in dynamics of $a_t$ and $v_t$,$\alpha = 3, \beta = 2$, and two shocks are bivariate normal distributed with correlation 0.1,

$$\omega_t, \gamma_t \sim \mathcal{N}(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0.1 \\ 0.1 & 1 \end{bmatrix})$$

The result are calculated under different pair of $(\eta_1, \eta_2)$: (-0.3, 0.7), (-0.1, 0.9), (0, 0.5), (0.2, 0.8), (0.5, 0.5), as Figure 1 shows.

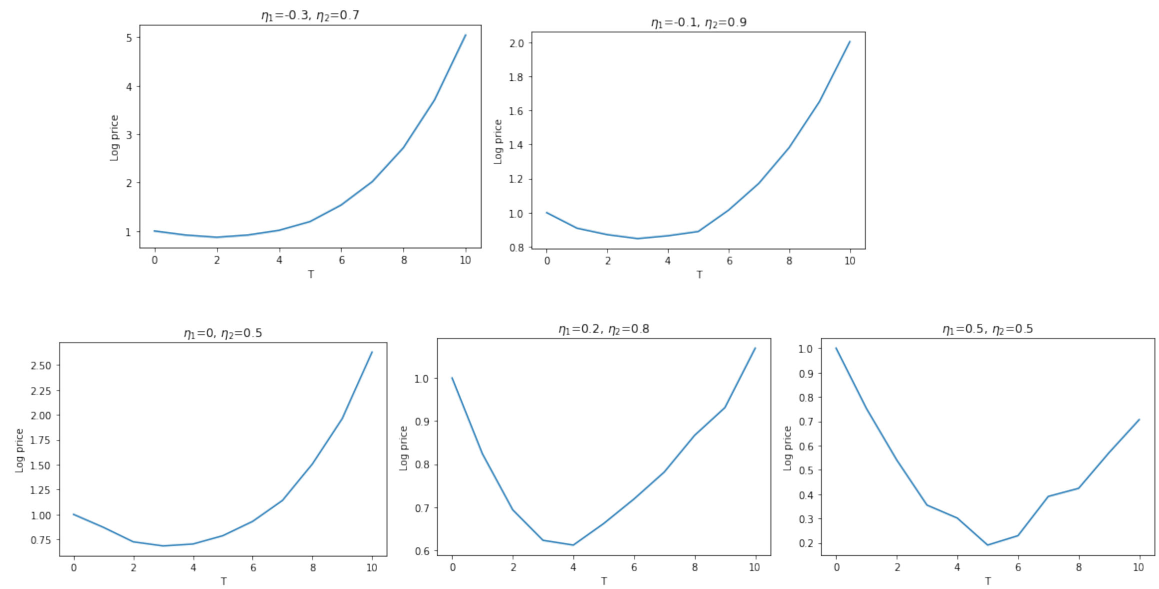


Figure 1: Log price with bivariate normal shocks

We see the the prices all decrease at first then increase for different $\eta$. This is because we assume there are shocks for each period. At the beginning, correlated shocks has large impact on the price, even larger than the growth rate of the value. However, in real market, exogenous shocks cannot happen for every period, and are more likely to behave as jump process.

In the next step, we assume two shocks has compound Poisson process with different rates. Since regulatory shock is more frequent technology shocks, we assume $\lambda_\omega = \frac{1}{2}$ and $\lambda_\gamma = \frac{1}{10}$. Then, one can simulate the price samples by 2 steps:

1. Generate the samples of the jump times $(\tau^k)_{k\geq 1}$ of $Poisson(\frac{1}{2})$ and $Poisson(\frac{1}{10})$ respectively
2. For each $k$, generate the size of jump from $\mathcal{N}(0,1)$ independently, and update the price $s_{\tau^k}$ with the jump.

Still, we simulate price under five different pair of $(\eta_1, \eta_2)$ as above. Figure 2 shows the result.

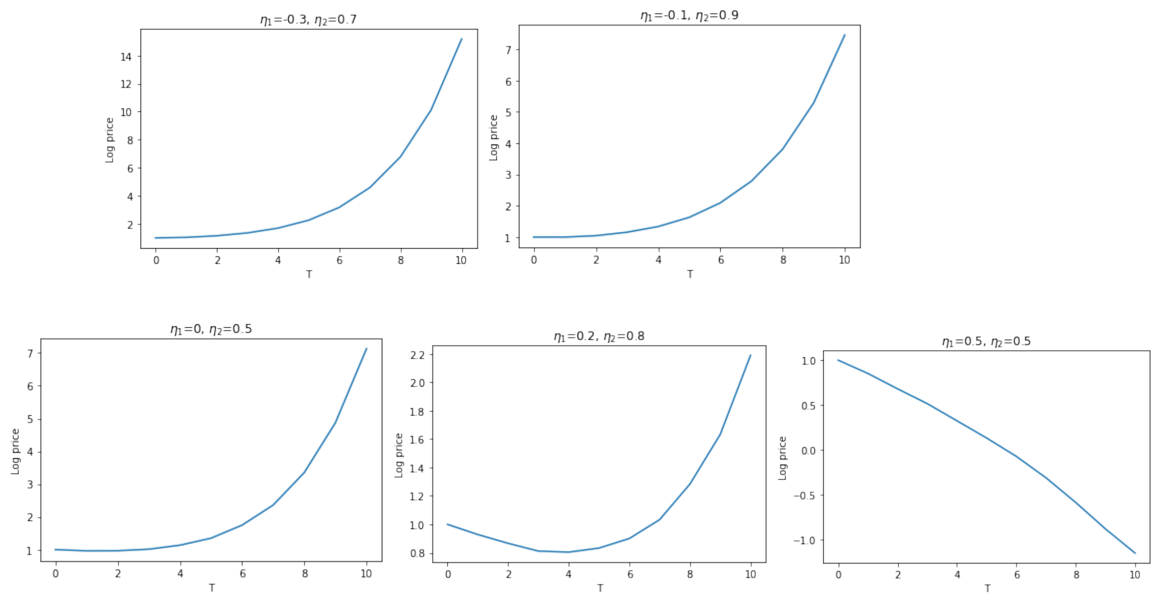


Figure 2: Log price with jumps

In this jump process case, we have more realistic result. For example, when $(\eta_1, \eta_2) = (-0.1, 0.9)$, the price of cryptocurrency is around 1750 USD after 10 years. Also, $\eta_1$ and $\eta_2$ has different effect on the price. When $\eta_1 > 0$, the price tends to decrease over time with higher $\eta_1$. When $\eta_1 < 0$, the price tends to increasing with lower $\eta_1$. When $\eta_2$ higher, the price tends to increasing over time.

# 3 A Speculators Dominating Market

## 3.1 Equilibrium Price

In the current cryptocurrency market, demand for speculation is far higher than the medium of exchange, as there is a limited number of commodities that could be purchased by paying cryptocurrency. Featured with speculators dominating, we can approximate the demand for cryptocurrency by the demand for speculation.

**Assumption 3 :** $K \cdot S_t^{\eta_1} \cdot A_t^{\eta_2}$ is larger than $\theta$ such that

$$\log(1 + K \cdot S_t^{\eta_1} \cdot A_t^{\eta_2}) \approx \log(K) + \eta_1 \log(S_t) + \eta_2 \log(A_t) \tag{17}$$

Under Assumption 3, equilibrium price $s_t$ satisfies

$$s_t + m_t = k + (\eta_1 - 1)s_t + \eta_2 a_t + g_t - v_t \tag{18}$$

**Proposition 3 :** In the speculator dominating market, the sensitive of equilibrium price attractiveness is

$$\frac{\partial s_t}{\partial a_t} = \frac{\eta_2}{2-\eta_1} > 0 \tag{19}$$

With the dynamic of $v_t$ in Equation (13), we can solve for $s_t$ recursively,

$$s_t = \frac{1}{2-\eta_1+\alpha}\mathbb{E}_t\sum_{j=0}^{\infty}\left(\frac{\alpha}{2-\eta_1+\alpha}\right)^j (k + \eta_2 a_{t+j} + g_{t+j} - m_{t+j} - \gamma_{t+j}) \tag{20}$$

From it, we notice the price is increasing with discounted value of future attractiveness.

Furthermore, we could choose the dynamic of $a_t$ as Equation (15). Under the equilibrium, the price follows

$$(2-\eta_1+\eta_2\beta+\alpha)\, s_t = (\eta_2\beta+\alpha)\mathbb{E}_t s_{t+1} + k + (1+\eta_2\theta)g_t - m_t + \eta_2\omega_t - \gamma_t \tag{21}$$

**Proposition 4 :** Solve Equation (16) recursively, we have an model-implied price for cryptocurrency as a function of

$$s_t = \frac{1}{2-\eta_1+\eta_2\beta+\alpha}\mathbb{E}_t\sum_{j=0}^{\infty}\left(\frac{\eta_2\beta+\alpha}{2-\eta_1+\eta_2\beta+\alpha}\right)^j (k + (1+\eta_2\theta)g_t - m_{t+j} + \eta_2\omega_{t+j} - \gamma_{t+j}) \tag{22}$$

Current size of the cryptocurrency economy and the total stock of the cryptocurrency as well as their expectation determine the current price of cryptocurrency.

Notice the discount factor $\frac{\eta_2\beta+\alpha}{2-\eta_1+\eta_2\beta+\alpha}$ in the price equation, one can consider the cryptocurrency as money account in the cryptocurrency market. Then the interest rate in the market is $\frac{2-\eta_1+\eta_2\beta+\alpha}{\eta_2\beta+\alpha} - 1$.

## 3.2 Numerical Illustration

With the price's sensitivity to attractiveness in Proposition 3, one can show how the sensitivity varies with different $\eta_1$ and $\eta_2$. If $\eta_1$ ranges from $-1$ to 1 and $\eta_2$ ranges from 0 to 2, the sensitivity of $s_t$ to $a_t$ is as Figure 3.

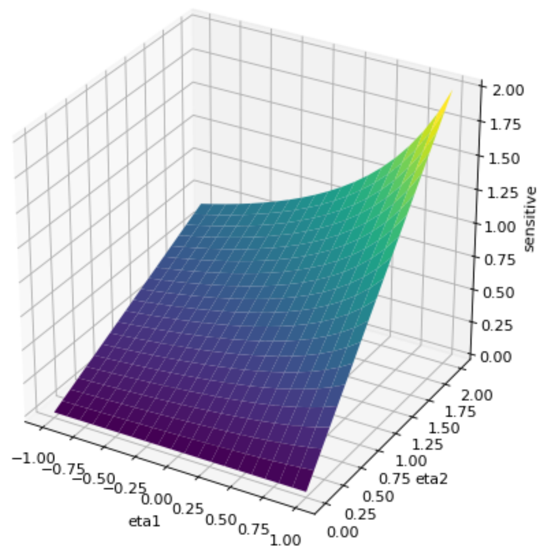


Figure 3: Price's sensitivity to attractiveness

Using Equation (21) and the parameters in Section 2.5, we simulate the price path under compound Poisson shocks.

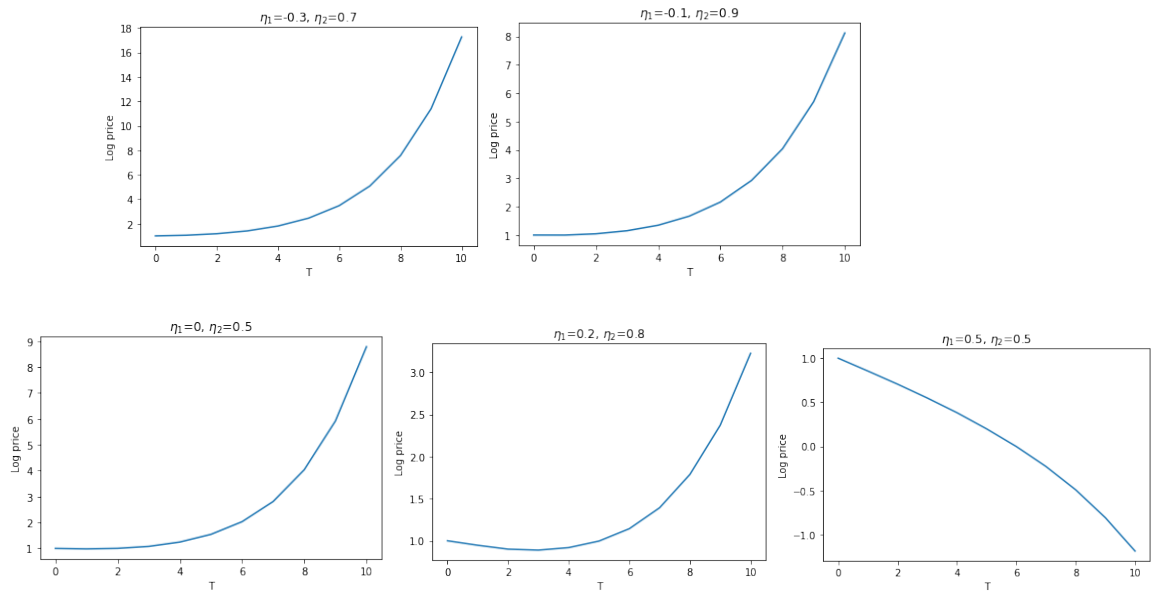


Figure 4: Log price with jumps in the speculators dominating market

The pattern is similar to Figure 2, except the prices grow faster in Figure 4. The impact of $\eta_1$ and $\eta_2$ is easier to analysis in the speculators dominating market, since $\eta_1$ and $\eta_2$ only enter the model via the growth rate $\frac{2-\eta_1+\eta_2\beta+\alpha}{\eta_2\beta+\alpha}$.

## 4 Future research

In this paper, we have an equilibrium model of cryptocurrency price and its sensitivity to attractiveness. For further research, one can be done is calibrating parameters in simulation part from market data of cryptocurrencies, like Bitcoin or ETH.

Also, empirical analysis can be implemented as in Jermann (2018), which proposed a linear relationship between price changes, transaction volumes changes, and money supply: $\Delta s_t = \alpha + \beta_1 \Delta\tau_t - \beta_2 \Delta m_t + \gamma_t \Delta$. Using similar regression, one can exam the relation between price changes, attractiveness changes and stock changes in our framework.

## References


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